

LEVEL N

N 1-10 : Arithmetic Sequences

The **general term** of an arithmetic sequence $\{a_n\}$ with 1st term a and common difference d is

$$a_n = a + (n - 1)d$$

Find the 1st term a and the general term of the arithmetic sequence $\{a_n\}$ whose common difference is -2 and 6th term is 8.

[Sol] $a_6 = a + 5 \cdot (-2) = 8$

$$\therefore a = 18$$

$$\text{Also, } a_n = 18 + (n-1) \cdot (-2) = -2n + 20$$

Find the 55th term of the arithmetic sequence $\{a_n\}$ whose 15th term is 77 and 42nd term is 239.

Let a be the 1st term and d be the common difference.

$$\begin{cases} a_{15} = a + 14d = 77 & \dots \textcircled{1} \\ a_{42} = a + 41d = 239 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $a = -7$, $d = 6$

$$\therefore a_{55} = -7 + 54 \cdot 6 = 317$$

Let S_n be the **sum** of an arithmetic sequence with 1st term a , common difference d , last term l and number of terms n .

$$S_n = \frac{1}{2}n(a + l) = \frac{1}{2}n[2a + (n - 1)d]$$

Find the general term of the arithmetic sequence $\{a_n\}$ whose sum of the first 5 terms is 100 and whose sum of the first 10 terms is 150.

Let a be the 1st term, d be the common difference and S_n be the sum of the first n terms.

$$\begin{cases} S_5 = \frac{1}{2} \cdot 5[2a + (5-1)d] = 100 & \dots \textcircled{1} \\ S_{10} = \frac{1}{2} \cdot 10[2a + (10-1)d] = 150 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $a = 24$, $d = -2$

$$\therefore a_n = 24 + (n-1) \cdot (-2) = -2n + 26$$

N 11-20 : Geometric Sequences

The **general term** of a geometric sequence $\{a_n\}$ with 1st term a and common ratio r is

$$a_n = ar^{n-1}$$

Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose 3rd term is 12 and whose sum of the first 3 terms is 21.

$$\begin{cases} ar^2 = 12 & \dots \textcircled{1} \\ a + ar + ar^2 = 21 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{2}$, $a(1 + r + r^2) = 21$

$$12(1 + r + r^2) = 21r^2 \quad \therefore r = -\frac{2}{3}, 2$$

(i) When $r = -\frac{2}{3}$, from $\textcircled{1}$, $a = 27$

(ii) When $r = 2$, from $\textcircled{1}$, $a = 3$

From (i) and (ii), $a = 27$, $r = -\frac{2}{3}$ or $a = 3$, $r = 2$

Let S_n be the **sum** of a geometric sequence with 1st term a , common ratio r and number of terms n .

$$\text{When } r \neq 1, S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$$

$$\text{When } r = 1, S_n = na$$

Given the geometric sequence whose 1st term is 6 and common ratio is -2 , find the value of n for which the sum of the first n terms is -30 .

Let S_n be the sum of the first n terms.

$$S_n = \frac{6[1 - (-2)^n]}{1 - (-2)} = -30$$

$$(-2)^n = 16$$

$$\therefore n = 4$$

Find the 1st term a and the common ratio r of the geometric sequence $\{a_n\}$ whose sum of the first 3 terms is 6 and whose sum of the first 6 terms is -42 . (The common ratio is a real number.)

Let S_n be the sum of the first n terms.

(i) When $r = 1$, $S_3 = 3a = 6$, $S_6 = 6a = -42$

No a can satisfy these conditions simultaneously.

(ii) When $r \neq 1$,

$$\begin{cases} S_3 = \frac{a(r^3-1)}{r-1} = 6 & \dots \textcircled{1} \\ S_6 = \frac{a(r^6-1)}{r-1} = -42 & \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$, $6(r^3+1) = -42$

$$\therefore r = -2$$

From $\textcircled{1}$, $a = 2$

From (i) and (ii), $a = 2$, $r = -2$

N 21-40 : Various Sequences

Summation Properties:

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$$

$$\sum_{k=1}^n ca_k = c \sum_{k=1}^n a_k \quad (\text{where } c \text{ is a constant})$$

Summation Formulas:

$$\sum_{k=1}^n k = \frac{1}{2}n(n+1), \quad \sum_{k=1}^n c = nc$$

$$\sum_{k=1}^n k^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{k=1}^n k^3 = \frac{1}{4}n^2(n+1)^2$$

$$\sum_{k=1}^n ar^{k-1} = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1} \quad (r \neq 1)$$

Examples:

$$\begin{aligned} & \sum_{k=1}^n (6k^2 + 2k + 4^{k+1}) \\ &= 6 \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k + \sum_{k=1}^n 16 \cdot 4^{k-1} \\ &= 6 \cdot \frac{1}{6}n(n+1)(2n+1) + 2 \cdot \frac{1}{2}n(n+1) + \frac{16(4^n-1)}{4-1} \\ &= n(n+1)[(2n+1)+1] + \frac{4^{n+2}-16}{3} \\ &= 2n(n+1)^2 + \frac{4^{n+2}-16}{3} \end{aligned}$$

$$\begin{aligned}
 & \sum_{k=1}^{2n} (2k^3 + 3k^2 + 6k^{-1}) \\
 &= 2 \sum_{k=1}^{2n} k^3 + 3 \sum_{k=1}^{2n} k^2 + \sum_{k=1}^{2n} 6k^{-1} \\
 &= 2 \left[\frac{1}{2} \cdot 2n(2n+1) \right]^2 + 3 \cdot \frac{1}{6} \cdot 2n(2n+1)(2 \cdot 2n+1) + \frac{6^{2n}-1}{6-1} \\
 &= n(2n+1)[2n(2n+1) + (4n+1)] + \frac{6^{2n}-1}{5} \\
 &= n(2n+1)(4n^2+6n+1) + \frac{6^{2n}-1}{5}
 \end{aligned}$$

Sequence of Differences and General term

Let $\{b_n\}$ be the sequence of differences of the sequence $\{a_n\}$. When $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k$$

2, 10, 24, 44, 70, 102, ...

Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is
8, 14, 20, 26, 32, ...

$$\therefore b_n = 8 + (n-1) \cdot 6 = 6n + 2$$

When $n \geq 2$, $a_n = a_1 + \sum_{k=1}^{n-1} (6k+2)$

$$= 2 + 6 \cdot \frac{1}{2} (n-1)n + 2(n-1)$$

$$= n(3n-1) \quad \dots \textcircled{1}$$

Since $a_1 = 2$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = n(3n-1)$$

3, 4, 1, 10, -17, 64, ...

Let $\{b_n\}$ be the sequence of differences of $\{a_n\}$. Then, $\{b_n\}$ is
1, -3, 9, -27, 81, ...

$$\therefore b_n = 1 \cdot (-3)^{n-1} = (-3)^{n-1}$$

When $n \geq 2$, $a_n = a_1 + \sum_{k=1}^{n-1} (-3)^{k-1}$

$$= 3 + \frac{1 - (-3)^{n-1}}{1 - (-3)}$$

$$= \frac{13 - (-3)^{n-1}}{4} \quad \dots \textcircled{1}$$

Since $a_1 = 3$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = \frac{13 - (-3)^{n-1}}{4}$$

Let S_n be the sum of the first n terms of the sequence $\{a_n\}$. The 1st term is $a_1 = S_1$.

When $n \geq 2$, $a_n = S_n - S_{n-1}$

$$S_n = n^2 + 4n$$

The 1st term a_1 is $a_1 = S_1 = 1^2 + 4 \cdot 1 = 5$.

When $n \geq 2$,

$$a_n = S_n - S_{n-1}$$

$$= (n^2 + 4n) - [(n-1)^2 + 4(n-1)]$$

$$= 2n + 3 \quad \dots \textcircled{1}$$

Since $a_1 = 5$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = 2n + 3$$

Variation: Each term in an arithmetic sequence $\{a_n\}$ multiplied by r^n or r^{n-1}

$$S = 1 \cdot 1 + 2 \cdot 3 + 3 \cdot 3^2 + 4 \cdot 3^3 + \dots + n \cdot 3^{n-1} \quad \dots \textcircled{1}$$

Multiplying both sides of $\textcircled{1}$ by 3,

$$3S = 1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \dots + (n-1) \cdot 3^{n-1} + n \cdot 3^n \quad \dots \textcircled{2}$$

$$\textcircled{1} - \textcircled{2}, \quad -2S = (1 + 3 + 3^2 + 3^3 + \dots + 3^{n-1}) - n \cdot 3^n$$

$$= \frac{3^n - 1}{3 - 1} - n \cdot 3^n$$

$$= -\frac{(2n-1) \cdot 3^n + 1}{2}$$

$$\therefore S = \frac{(2n-1) \cdot 3^n + 1}{4}$$

Variation: Telescoping sum

$$S = \frac{3}{1 \cdot 4} + \frac{3}{4 \cdot 7} + \frac{3}{7 \cdot 10} + \dots + \frac{3}{(3n-2)(3n+1)}$$

$$\text{Since } \frac{3}{(3k-2)(3k+1)} = \frac{1}{3k-2} - \frac{1}{3k+1}$$

Therefore,

$$S = \left(\frac{1}{1} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{10} \right) + \dots + \left(\frac{1}{3n-2} - \frac{1}{3n+1} \right)$$

$$= 1 - \frac{1}{3n+1} = \frac{3n}{3n+1}$$

N 41-50 : Recurrence Relations

The sequence $\{a_n\}$ defined by $a_{n+1} = a_n + d$ is an arithmetic sequence with common difference d .

The sequence $\{a_n\}$ defined by $a_{n+1} = ra_n$ is a geometric sequence with common ratio r .

Given sequence $\{a_n\}$ defined by $a_{n+1} = a_n + b_n$, then
 $a_n = a_1 + \sum_{k=1}^{n-1} b_k$ for $n \geq 2$.

$$a_1 = 4, \quad a_{n+1} = a_n + 4n^3$$

$$a_{n+1} - a_n = 4n^3$$

Since the general term of the sequence of differences of $\{a_n\}$ is $4n^3$, when $n \geq 2$,

$$a_n = a_1 + \sum_{k=1}^{n-1} 4k^3 = 4 + 4 \left[\frac{1}{2} (n-1)n \right]^2 \quad \sum_{k=1}^n k^3 = \left[\frac{1}{2} n(n+1) \right]^2$$

$$= n^4 - 2n^3 + n^2 + 4 \quad \dots \textcircled{1}$$

Since $a_1 = 4$, $\textcircled{1}$ is also true when $n = 1$.

$$\therefore a_n = n^4 - 2n^3 + n^2 + 4$$

The recurrence relation $a_{n+1} = pa_n + q$ can be rearranged into $a_{n+1} - x = p(a_n - x)$ by using x which satisfies $x = px + q$. Letting $b_n = a_n - x$ we have that $\{b_n\}$ is a geometric sequence.

$$a_1 = 4, \quad a_{n+1} = 2a_n + 1$$

$$\text{Since } a_{n+1} = 2a_n + 1, \quad a_{n+1} + 1 = 2(a_n + 1)$$

Let $b_n = a_n + 1$.

$$\begin{cases} b_{n+1} = 2b_n & \dots \textcircled{1} \\ b_1 = a_1 + 1 = 5 & \dots \textcircled{2} \end{cases}$$

$$\text{From } \textcircled{1} \text{ and } \textcircled{2}, \quad b_n = 5 \cdot 2^{n-1} \quad \therefore a_n = b_n - 1 = 5 \cdot 2^{n-1} - 1$$

$$\begin{aligned}
 & \text{Replacing both} \\
 & a_{n+1} \text{ and } a_n \text{ with } x, \\
 & x = 2x + 1 \\
 & \therefore x = -1
 \end{aligned}$$

The recurrence relation $a_{n+2} + pa_{n+1} + qa_n = 0$ can be rearranged into $a_{n+2} - \alpha a_{n+1} = \beta(a_{n+1} - \alpha a_n)$ by using two solutions α and β of quadratic equation $x^2 + px + q = 0$. The sequence $\{a_{n+1} - \alpha a_n\}$ is a geometric sequence.

$$a_1 = 2, a_2 = 1, a_{n+2} = a_{n+1} + 6a_n$$

$$\text{Since } a_{n+2} = a_{n+1} + 6a_n, a_{n+2} - a_{n+1} - 6a_n = 0 \dots \textcircled{1}$$

Rearranging $\textcircled{1}$,

$$\begin{cases} a_{n+2} + 2a_{n+1} = 3(a_{n+1} + 2a_n) & \dots \textcircled{2} \\ a_{n+2} - 3a_{n+1} = -2(a_{n+1} - 3a_n) & \dots \textcircled{3} \end{cases}$$

From $\textcircled{2}$, the sequence $\{a_{n+1} + 2a_n\}$ is the geometric sequence with 1st term $a_2 + 2a_1 = 5$ and common ratio 3.

$$\therefore a_{n+1} + 2a_n = 5 \cdot 3^{n-1} \dots \textcircled{4}$$

From $\textcircled{3}$, the sequence $\{a_{n+1} - 3a_n\}$ is the geometric sequence with 1st term $a_2 - 3a_1 = -5$ and common ratio -2 .

$$\therefore a_{n+1} - 3a_n = -5(-2)^{n-1} \dots \textcircled{5}$$

$$\text{From } \textcircled{4} \text{ and } \textcircled{5}, a_n = 3^{n-1} + (-2)^{n-1}$$

Note: Refer to the worksheets for more variations.

N 51-60 : Mathematical Induction

To prove that a proposition P is true for all natural numbers n by mathematical induction,

- 1) Prove that P is true when $n = 1$ (base number).
- 2) Assume that P is true when $n = k$.
- 3) Prove that P is also true when $n = k + 1$.

Tips: When proving the statement for $n = k + 1$, try to rearrange the expression so that it is possible to use the statement for $n = k$.

To prove an **inequality**, take the difference LHS – RHS and show that it is > 0 or < 0 when $n = k + 1$.

Refer to the worksheets for elaborated examples.

N 61-70 : Infinite Sequences

An infinite sequence may:

- **converge** $\lim_{n \rightarrow \infty} a_n = \alpha$

- **diverge** $\lim_{n \rightarrow \infty} a_n = \pm \infty$ or $\{a_n\}$ oscillates

Properties of Limits of Sequences

When the sequences $\{a_n\}$ and $\{b_n\}$ converge, where $\lim_{n \rightarrow \infty} a_n = \alpha$ and $\lim_{n \rightarrow \infty} b_n = \beta$,

1) $\lim_{n \rightarrow \infty} ka_n = k\alpha$ (k is a constant)

2) $\lim_{n \rightarrow \infty} a_n \pm b_n = \alpha \pm \beta$

3) $\lim_{n \rightarrow \infty} a_n b_n = \alpha \beta$

4) $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\alpha}{\beta}$ ($\beta \neq 0$)

There are several **techniques** to evaluate limits.

Technique 1: Divide the numerator and denominator by the term with the highest power in denominator

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{-2n^2(n+2)}{2n^3-5} &= \lim_{n \rightarrow \infty} \frac{-2n^3-4n^2}{2n^3-5} = \lim_{n \rightarrow \infty} \frac{-2-\frac{4}{n}}{2-\frac{5}{n^3}} = -1 \\ \lim_{n \rightarrow \infty} \frac{5+n+n^5}{n(4-3n^2)} &= \lim_{n \rightarrow \infty} \frac{5+n+n^5}{4n-3n^3} = \lim_{n \rightarrow \infty} \frac{\frac{5}{n^3}+\frac{1}{n^2}+n^2}{\frac{4}{n^2}-3} = -\infty \quad \left(\lim_{n \rightarrow \infty} \left(\frac{5}{n^3}+\frac{1}{n^2}+n^2 \right) = \infty \right. \\ &\quad \left. \lim_{n \rightarrow \infty} \left(\frac{4}{n^2}-3 \right) = -3 \right) \\ \lim_{n \rightarrow \infty} \frac{\sqrt{n^2+1}}{2n} &= \lim_{n \rightarrow \infty} \frac{\sqrt{1+\frac{1}{n^2}}}{2} = \frac{1}{2} \quad \left(\frac{\sqrt{n^2+1}}{n} = \sqrt{\frac{n^2+1}{n^2}} = \sqrt{\frac{n^2}{n^2}+\frac{1}{n^2}} \right) \\ \lim_{n \rightarrow \infty} [\log_3(2n^2-1) - \log_3(6n^2+2)] &= \lim_{n \rightarrow \infty} \log_3 \frac{2n^2-1}{6n^2+2} \\ &= \lim_{n \rightarrow \infty} \log_3 \frac{2-\frac{1}{n^2}}{6+\frac{2}{n^2}} \\ &= \log_3 \frac{1}{3} = -1 \end{aligned}$$

Technique 2: Factor out the dominant term

$$\begin{aligned} \lim_{n \rightarrow \infty} (n^2-3n) &= \lim_{n \rightarrow \infty} n^2 \left(1 - \frac{3}{n} \right) = \infty \quad \left(\text{Taking out the term with the highest power} \right) \\ \lim_{n \rightarrow \infty} (2n^3-3n^2+4) &= \lim_{n \rightarrow \infty} n^3 \left(2 - \frac{3}{n} + \frac{4}{n^3} \right) = \infty \\ \lim_{n \rightarrow \infty} (4n^2-2n^3) &= \lim_{n \rightarrow \infty} n^3 \left(\frac{4}{n} - 2 \right) = -\infty \end{aligned}$$

Technique 3: Multiply the numerator and the denominator by conjugate surds

$$\begin{aligned} \lim_{n \rightarrow \infty} (\sqrt{n^2+3n}-n) &= \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2+3n}-n)(\sqrt{n^2+3n}+n)}{\sqrt{n^2+3n}+n} \\ &= \lim_{n \rightarrow \infty} \frac{3n}{\sqrt{n^2+3n}+n} \\ &= \lim_{n \rightarrow \infty} \frac{3}{\sqrt{1+\frac{3}{n}}+1} \\ &= \frac{3}{2} \end{aligned}$$

Considering $\sqrt{n^2+3n}-n = \frac{n^2+3n-n^2}{\sqrt{n^2+3n}+n} = \frac{3n}{\sqrt{n^2+3n}+n}$, then multiplying the numerator and the denominator by $\sqrt{n^2+3n}+n$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5}-\sqrt{n+3})}{\sqrt{n+2}-\sqrt{n}} &= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+5}-\sqrt{n+3})(\sqrt{n+5}+\sqrt{n+3})(\sqrt{n+2}+\sqrt{n})}{(\sqrt{n+2}-\sqrt{n})(\sqrt{n+2}+\sqrt{n})(\sqrt{n+5}+\sqrt{n+3})} \\ &= \lim_{n \rightarrow \infty} \frac{2(\sqrt{n+2}+\sqrt{n})}{\sqrt{n+5}+\sqrt{n+3}} \\ &= \lim_{n \rightarrow \infty} \frac{2\left(\sqrt{1+\frac{2}{n}}+1\right)}{\sqrt{1+\frac{5}{n}}+\sqrt{1+\frac{3}{n}}} = 2 \end{aligned}$$

Technique 4: The Squeeze Theorem

For all n , when $a_n \leq b_n$,

if $\lim_{n \rightarrow \infty} a_n = \alpha$ and $\lim_{n \rightarrow \infty} b_n = \beta$, then $\alpha \leq \beta$

if $\lim_{n \rightarrow \infty} a_n = \infty$, then $\lim_{n \rightarrow \infty} b_n = \infty$

For all n , when $a_n \leq c_n \leq b_n$,

if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \alpha$, then $\lim_{n \rightarrow \infty} c_n = \alpha$ (*)

(*) is known as the **Squeeze Theorem**.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sin \frac{n\pi}{2}$$

$$\text{Since } -1 \leq \sin \frac{n\pi}{2} \leq 1,$$

$$-\frac{1}{n} \leq \frac{1}{n} \sin \frac{n\pi}{2} \leq \frac{1}{n}$$

$$\text{Then, since } \lim_{n \rightarrow \infty} \left(-\frac{1}{n}\right) = 0 \text{ and } \lim_{n \rightarrow \infty} \frac{1}{n} = 0,$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sin \frac{n\pi}{2} = 0$$

$$\lim_{n \rightarrow \infty} \frac{n+1}{n^2} \cos n\theta \quad (\theta \text{ is a constant})$$

$$\text{Since } -1 \leq \cos n\theta \leq 1,$$

$$-\frac{n+1}{n^2} \leq \frac{n+1}{n^2} \cos n\theta \leq \frac{n+1}{n^2}$$

$$\text{Then, since } \lim_{n \rightarrow \infty} \left(-\frac{n+1}{n^2}\right) = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n^2}\right) = 0 \text{ and}$$

$$\lim_{n \rightarrow \infty} \frac{n+1}{n^2} = \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n^2}\right) = 0,$$

$$\lim_{n \rightarrow \infty} \frac{n+1}{n^2} \cos n\theta = 0$$

N 71-80 : Infinite Geometric Sequences

Limit of an Infinite Geometric Sequence $\{r_n\}$

When $r < 1$, $\lim_{n \rightarrow \infty} r^n = 0$... Diverges

When $r = 1$, $\lim_{n \rightarrow \infty} r^n = 1$... Converges

When $|r| < 1$, $\lim_{n \rightarrow \infty} r^n = 0$... Converges

When $r \leq -1$, Oscillates ... Diverges

Technique 1: Divide the numerator and the denominator by the term with the largest absolute value of the base in the denominator

$$\lim_{n \rightarrow \infty} \frac{(-2)^n + 2 \cdot 3^n}{3^n + 1} = \lim_{n \rightarrow \infty} \frac{\left(-\frac{2}{3}\right)^n + 2}{1 + \left(\frac{1}{3}\right)^n} = 2$$

$$\lim_{n \rightarrow \infty} \frac{-5^n + 3^{n+1}}{3^n + 4^n} = \lim_{n \rightarrow \infty} \frac{-\left(\frac{5}{4}\right)^n + 3\left(\frac{3}{4}\right)^n}{\left(\frac{3}{4}\right)^n + 1} = -\infty$$

Technique 2: Factor out the term with the largest absolute value of the base

$$\lim_{n \rightarrow \infty} (4^n - 3^n) = \lim_{n \rightarrow \infty} 4^n \left[1 - \left(\frac{3}{4}\right)^n\right] = \infty$$

$$\lim_{n \rightarrow \infty} (3^n - 5^n) = \lim_{n \rightarrow \infty} 5^n \left[\left(\frac{3}{5}\right)^n - 1\right] = -\infty$$

Variation 1: Consider the different cases of values of the geometric ratio

$$\lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1}$$

$$\text{When } |r| < 1, \quad \lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1} = \frac{0 - 1}{0 + 1} = -1$$

$$\text{When } r = 1, \quad \lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1} = \frac{1 - 1}{1 + 1} = 0$$

$$\text{When } r = -1, \quad \lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1} = \frac{-1 - 1}{1 + 1} = -1$$

$$\text{When } |r| > 1, \quad \lim_{n \rightarrow \infty} \frac{r^{2n+1} - 1}{r^{2n} + 1} = \lim_{n \rightarrow \infty} \frac{r - \left(\frac{1}{r}\right)^{2n}}{1 + \left(\frac{1}{r}\right)^{2n}} = \frac{r - 0}{1 + 0} = r$$

Variation 2: A geometric sequence converges when if and only if $-1 < r \leq 1$.

Find the range of values of x for which the sequence $\left\{\left(\frac{x+1}{3}\right)^n\right\}$ converges. Then, state the limit values.

$$\text{Since the common ratio is } \frac{x+1}{3}, \quad -1 < \frac{x+1}{3} \leq 1$$

$$\therefore -4 < x \leq 2$$

Also, the limit values are,

$$\text{when } -4 < x < 2, \quad \lim_{n \rightarrow \infty} \left(\frac{x+1}{3}\right)^n = 0 \text{ and}$$

$$\text{when } x = 2, \quad \lim_{n \rightarrow \infty} \left(\frac{x+1}{3}\right)^n = 1$$

N 81-90 : Infinite Geometric Series

Given an infinite geometric series $a + ar + ar^2 + \dots + ar^{n-1} + \dots$, the following is true.

When $a \neq 0$,

if $|r| < 1$, then the series converges to $\frac{a}{1-r}$

if $|r| \geq 1$, then the series diverges

When $a = 0$, the series converges to 0.

$$(\sqrt{3}-1) - 2(2-\sqrt{3}) + 2(3\sqrt{3}-5) - 4(7-4\sqrt{3}) + \dots$$

The 1st term is $\sqrt{3}-1$ and the common ratio is $-\sqrt{3}+1$.

Since $|-\sqrt{3}+1| < 1$, the series **converges**.

$$\therefore S = \frac{\sqrt{3}-1}{1-(-\sqrt{3}+1)} = \frac{\sqrt{3}-1}{\sqrt{3}} = \frac{3-\sqrt{3}}{3}$$

When $a \neq 0$, if $|r| < 1$, then the infinite geometric series converges.

Find the range of values of a real number x for which the following infinite geometric series converges. Then, find the sum S .

$$2 + 2(x^2 - 3) + 2(x^2 - 3)^2 + \dots$$

Since the common ratio is $x^2 - 3$, $-1 < x^2 - 3 < 1$

So, $-1 < x^2 - 3 \dots \textcircled{1}$ and also $x^2 - 3 < 1 \dots \textcircled{2}$

From $\textcircled{1}$, $x^2 > 2$; therefore, $x < -\sqrt{2}$, $\sqrt{2} < x \dots \textcircled{3}$

From $\textcircled{2}$, $x^2 < 4$; therefore, $-2 < x < 2 \dots \textcircled{4}$

From $\textcircled{3}$ and $\textcircled{4}$, the range of values of x is

$$-2 < x < -\sqrt{2}, \quad \sqrt{2} < x < 2$$

$$\text{Also, } S = \frac{2}{1 - (x^2 - 3)} = \frac{2}{4 - x^2}$$

Some other variations/ interesting applications:

Given the infinite geometric series whose sum is 9 and 2nd term is -4, find the 1st term a and the common ratio r .

Since the sum is 9, $a \neq 0$, $-1 < r < 1$ Since the sum is 9, the infinite geometric series converges.

$$\begin{cases} \frac{a}{1-r} = 9 \dots \textcircled{1} \\ ar = -4 \dots \textcircled{2} \end{cases}$$

From $\textcircled{1}$ and $\textcircled{2}$,

$$9(1-r) \cdot r = -4$$

$$9r^2 - 9r - 4 = 0$$

$$(3r+1)(3r-4) = 0$$

From $\textcircled{1}$, $a = 9(1-r)$
Substituting this into $\textcircled{2}$

$$r = -\frac{1}{3}, \quad \frac{4}{3}$$

Since $-1 < r < 1$, $r = -\frac{1}{3}$

From $\textcircled{2}$, $a = 12$

$$0.\dot{3}0\dot{6} = 0.306 + 0.000306 + 0.00000306 + \dots$$

$0.\dot{3}0\dot{6}$ is the infinite geometric series with 1st term 0.306 and common ratio 0.001. Since $|0.001| < 1$, it converges.

$$\therefore 0.\dot{3}0\dot{6} = \frac{0.306}{1 - 0.001} = \frac{34}{111}$$

N 91-100 : Infinite Series

To determine whether an infinite series converges, first find the **partial sum** of the first n terms of the series, and then find the **limit** of the sum.

Key idea: Use standard tricks to obtain a telescoping sum, and then apply method of differences.

Technique 1: Multiply both the numerator and the denominator by the conjugate of the denominator

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} = \frac{\sqrt{n+1} - \sqrt{n}}{(\sqrt{n+1} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})} = \frac{\sqrt{n+1} - \sqrt{n}}{n+1 - n} = \sqrt{n+1} - \sqrt{n}$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) + \dots + (\sqrt{n+1} - \sqrt{n}) = -1 + \sqrt{n+1}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} (-1 + \sqrt{n+1}) = \infty$$

Thus, the series diverges.

Technique 2: Partial fraction decomposition

$$\frac{4}{1 \cdot 5} + \frac{4}{5 \cdot 9} + \frac{4}{9 \cdot 13} + \dots + \frac{4}{(4n-3)(4n+1)} + \dots$$

$$\text{Let } \frac{4}{(4n-3)(4n+1)} = \frac{a}{4n-3} - \frac{b}{4n+1} \quad \therefore a=1, \quad b=1$$

Therefore, let S_n be the partial sum of the first n terms.

$$S_n = \left(\frac{1}{1} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right) + \dots + \left(\frac{1}{4n-3} - \frac{1}{4n+1}\right) = 1 - \frac{1}{4n+1}$$

$$\therefore \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{4n+1}\right) = 1$$

Thus, the series **converges** and the sum is 1.

Technique 3: Consider partial sum of the first odd number of terms and the first even number of terms

$$\frac{1}{2} - \frac{2}{3} + \frac{2}{3} - \frac{3}{4} + \frac{3}{4} - \dots - \frac{n}{n+1} + \frac{n}{n+1} - \frac{n+1}{n+2} + \dots$$

Let S_n be the partial sum of the first n terms and m be a natural number.

(i) When $n = 2m - 1$,

$$S_n = S_{2m-1}$$

$$= \frac{1}{2} + \left(-\frac{2}{3} + \frac{2}{3}\right) + \left(-\frac{3}{4} + \frac{3}{4}\right) + \dots + \left(-\frac{m}{m+1} + \frac{m}{m+1}\right) = \frac{1}{2}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m-1} = \frac{1}{2}$$

$$\begin{aligned} S_1 &= \frac{1}{2} \\ S_3 &= \frac{1}{2} + \left(-\frac{2}{3} + \frac{2}{3}\right) \\ S_5 &= \frac{1}{2} + \left(-\frac{2}{3} + \frac{2}{3}\right) + \left(-\frac{3}{4} + \frac{3}{4}\right) \\ &\vdots \end{aligned}$$

(ii) When $n = 2m$,

$$S_n = S_{2m}$$

$$= S_{2m-1} + \left(-\frac{m+1}{m+2}\right)$$

$$S_{2m} = S_{2m-1} + (\text{the } 2m^{\text{th}} \text{ term})$$

$$\text{From (i), } \lim_{m \rightarrow \infty} S_{2m-1} = \frac{1}{2}$$

$$\therefore \lim_{m \rightarrow \infty} S_{2m} = \lim_{m \rightarrow \infty} \left(S_{2m-1} - \frac{1 + \frac{1}{m}}{1 + \frac{2}{m}}\right) = \frac{1}{2}$$

From (i) and (ii), $\lim_{m \rightarrow \infty} S_{2m-1} \neq \lim_{m \rightarrow \infty} S_{2m}$

Therefore, the series diverges.

(S_n) oscillates and diverges.

Properties of Infinite Series

When the infinite series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, where $\sum_{n=1}^{\infty} a_n = S$ and $\sum_{n=1}^{\infty} b_n = T$, the following properties are true:

$$\sum_{n=1}^{\infty} k a_n = kS \quad (k \text{ is a constant})$$

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = S \pm T$$

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{5}{4^n}\right) &= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{5}{4^n} \\ &= \frac{1}{1 - \frac{1}{2}} - \frac{\frac{5}{4}}{1 - \frac{1}{4}} \\ &= 2 - \frac{5}{3} = \frac{1}{3} \end{aligned} \quad \begin{aligned} \sum_{n=1}^{\infty} \left(\frac{2}{3^n} - \frac{1}{6^{n-1}}\right) &= \sum_{n=1}^{\infty} \frac{2}{3^n} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} \\ &= \frac{\frac{2}{3}}{1 - \frac{1}{3}} - \frac{1}{1 - \frac{1}{6}} \\ &= 1 - \frac{6}{5} = -\frac{1}{5} \end{aligned}$$

Important Theorem

If the infinite series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$. If the sequence $\{a_n\}$ does not converge to 0, then the infinite series $\sum_{n=1}^{\infty} a_n$ diverges.

N 101-120 : Limits of Functions

Properties of Infinite Series

If $\lim_{x \rightarrow a} f(x) = \alpha$ and $\lim_{x \rightarrow a} g(x) = \beta$, then

$$\lim_{x \rightarrow a} kf(x) = k\alpha \quad (k \text{ is a constant})$$

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \alpha \pm \beta$$

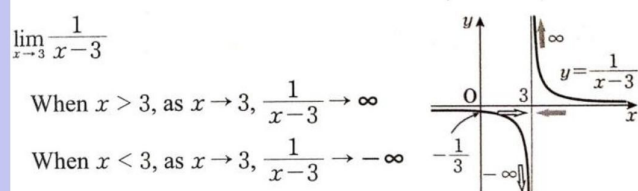
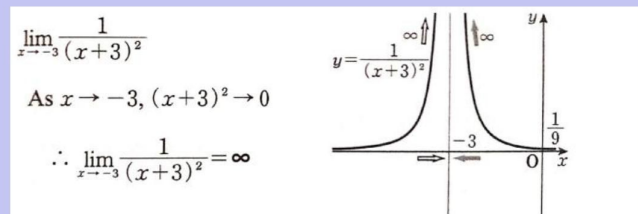
$$\lim_{x \rightarrow a} f(x)g(x) = \alpha\beta$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\alpha}{\beta} \quad (\beta \neq 0)$$

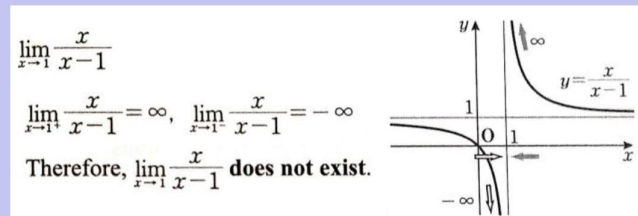
$$\lim_{x \rightarrow -2} \frac{x^3 + 8}{x^2 - x - 6} = \lim_{x \rightarrow -2} \frac{(x+2)(x^2 - 2x + 4)}{(x+2)(x-3)} = \lim_{x \rightarrow -2} \frac{x^2 - 2x + 4}{x-3} = -\frac{12}{5}$$

$$\lim_{x \rightarrow 1} \frac{x^2 - 4x + 3}{x^3 - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x-3)}{(x-1)(x^2 + x + 1)} = \lim_{x \rightarrow 1} \frac{x-3}{x^2 + x + 1} = -\frac{2}{3}$$

To evaluate limits of rational functions (near the asymptote), we need to draw graphs.



Definition: If $\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x)$, then $\lim_{x \rightarrow a} f(x)$ does not exist.



Techniques for evaluating limits of functions as x approaches infinity are similar to that of sequences (check N61-80).

$$\lim_{x \rightarrow \infty} \frac{2x^2 - x - 6}{3x^2 - 2x - 8} = \lim_{x \rightarrow \infty} \frac{2 - \frac{1}{x} - \frac{6}{x^2}}{3 - \frac{2}{x} - \frac{8}{x^2}} = \frac{2}{3}$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x+1} + \sqrt{3x-1}}{\sqrt{x-1}} = \lim_{x \rightarrow \infty} \frac{\sqrt{1 + \frac{1}{x}} + \sqrt{3 - \frac{1}{x}}}{\sqrt{1 - \frac{1}{x}}} = 1 + \sqrt{3} \quad \leftarrow \text{Dividing the numerator and the denominator by } \sqrt{x}$$

$$\lim_{x \rightarrow \infty} (x^2 - 3x + 2) = \lim_{x \rightarrow \infty} x^2 \left(1 - \frac{3}{x} + \frac{2}{x^2}\right) = \infty \quad \leftarrow \text{Taking out the term with the highest power}$$

$$\begin{aligned} \lim_{x \rightarrow \infty} (2x+1 - \sqrt{4x^2+2x+1}) &= \lim_{x \rightarrow \infty} \frac{(2x+1 - \sqrt{4x^2+2x+1})(2x+1 + \sqrt{4x^2+2x+1})}{2x+1 + \sqrt{4x^2+2x+1}} \\ &= \lim_{x \rightarrow \infty} \frac{2x}{2x+1 + \sqrt{4x^2+2x+1}} \\ &= \lim_{x \rightarrow \infty} \frac{2}{2 + \frac{1}{x} + \sqrt{4 + \frac{2}{x} + \frac{1}{x^2}}} = \frac{1}{2} \end{aligned}$$

$$\lim_{x \rightarrow \infty} (10^x - 3^x) = \lim_{x \rightarrow \infty} 10^x \left[1 - \left(\frac{3}{10}\right)^x\right] = \infty$$

$$\lim_{x \rightarrow \infty} (2^x - 5^x) = \lim_{x \rightarrow \infty} 5^x \left[\left(\frac{2}{5}\right)^x - 1\right] = -\infty \quad \leftarrow \begin{array}{l} \lim_{x \rightarrow \infty} 5^x = \infty \\ \lim_{x \rightarrow \infty} \left[\left(\frac{2}{5}\right)^x - 1\right] = -1 \end{array}$$

$$\lim_{x \rightarrow \infty} \frac{4^x - 1}{4^x + 1} = \lim_{x \rightarrow \infty} \frac{1 - \left(\frac{1}{4}\right)^x}{1 + \left(\frac{1}{4}\right)^x} = 1$$

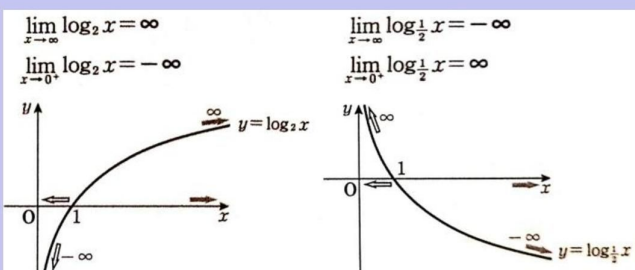
For limits that tend to negative infinity, we usually introduce a transformation of variable.

$$\lim_{x \rightarrow -\infty} \frac{4x^2 - x + 2}{2x - 3}$$

Let $x = -t$. As $x \rightarrow -\infty$, $t \rightarrow \infty$; therefore,

$$\lim_{x \rightarrow -\infty} \frac{4x^2 - x + 2}{2x - 3} = \lim_{t \rightarrow \infty} \frac{4t^2 + t + 2}{-2t - 3} = \lim_{t \rightarrow \infty} \frac{4t + 1 + \frac{2}{t}}{-2 - \frac{3}{t}} = -\infty$$

Recall important properties of logarithms. (Refer to the worksheets for more variation)



Theorem: If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \alpha < \infty$ and $\lim_{x \rightarrow a} g(x) = 0$, then we must have $\lim_{x \rightarrow a} f(x) = 0$.

N 121-130 : Limits of Trigonometric Functions

Main formulas: (can be used without proof)

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

Important theorems:

(1) For all x close to a , when $f(x) \leq h(x) \leq g(x)$, if $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = \alpha$, then $\lim_{x \rightarrow a} h(x) = \alpha$.

(2) If $\lim_{x \rightarrow a} |f(x)| = 0$, then $\lim_{x \rightarrow a} f(x) = 0$.

(1) is known as the **Squeeze Theorem**, (2) is known as the **Absolute Value Theorem**.

****Note that (2) is only true when the limit is zero.**

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{2x}{\sin 3x} &= \lim_{x \rightarrow 0} \frac{3x}{\sin 3x} \cdot \frac{2}{3} \\ &= \lim_{x \rightarrow 0} \frac{1}{\frac{\sin 3x}{3x}} \cdot \frac{2}{3} \\ &= \frac{1}{1} \cdot \frac{2}{3} = \frac{2}{3}\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 3x}{\tan 2x} &= \lim_{x \rightarrow 0} \frac{\sin 3x}{\frac{\sin 2x}{\cos 2x}} \\ &= \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 2x} \cdot \frac{\cos 2x}{1} \\ &= \frac{1 \cdot 3}{1 \cdot 2} = \frac{3}{2}\end{aligned}$$

Sometimes, we need to multiply the numerator and the denominator by an identified conjugate

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x \sin x}{1 - \cos x} &= \lim_{x \rightarrow 0} \frac{x \sin x (1 + \cos x)}{(1 - \cos x)(1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{x \sin x (1 + \cos x)}{\sin^2 x} \\ &= \lim_{x \rightarrow 0} \frac{1 + \cos x}{\frac{\sin x}{x}} = 2\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{2 \tan x}{\sqrt{1+x}-1} &= \lim_{x \rightarrow 0} \frac{2 \sin x (\sqrt{1+x}+1)}{\cos x (\sqrt{1+x}-1)(\sqrt{1+x}+1)} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin x (\sqrt{1+x}+1)}{x \cos x} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{2(\sqrt{1+x}+1)}{\cos x} = 4\end{aligned}$$

Transformation of variable is sometimes necessary

$$\begin{aligned}\lim_{x \rightarrow \infty} x \sin \frac{1}{x} & \quad \lim_{x \rightarrow \frac{\pi}{4}} \frac{2\pi - 8x}{\sin(4x - \pi)} \\ \text{Let } \frac{1}{x} = \theta. \text{ As } x \rightarrow \infty, \theta \rightarrow 0^+ & \quad \text{Let } x - \frac{\pi}{4} = \theta. x = \theta + \frac{\pi}{4}. \text{ As } x \rightarrow \frac{\pi}{4}, \theta \rightarrow 0 \\ \therefore \lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{\theta \rightarrow 0^+} \frac{1}{\theta} \sin \theta & \quad \therefore \lim_{x \rightarrow \frac{\pi}{4}} \frac{2\pi - 8x}{\sin(4x - \pi)} = \lim_{\theta \rightarrow 0} \frac{-8\theta}{\sin 4\theta} \\ & \quad = \lim_{\theta \rightarrow 0} \frac{-2}{\frac{\sin 4\theta}{4\theta}} = -2\end{aligned}$$

Application of the Absolute Value Theorem

$$\lim_{x \rightarrow \infty} \frac{\cos x}{x}$$

Since $0 \leq |\cos x| \leq 1$, $0 \leq \left| \frac{\cos x}{x} \right| = \left| \frac{1}{x} \right| |\cos x| \leq \left| \frac{1}{x} \right|$

Then, since $\lim_{x \rightarrow \infty} \left| \frac{1}{x} \right| = 0$, $\lim_{x \rightarrow \infty} \left| \frac{\cos x}{x} \right| = 0$

$\therefore \lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0$

N 131-140 : Continuous & Discontinuous Functions

A function $f(x)$ is said to be continuous at $x = a$ if $f(x)$ satisfies the following three conditions:

- $\lim_{x \rightarrow a} f(x)$ exists.
- $f(a)$ exists and is finite.
- $\lim_{x \rightarrow a} f(x) = f(a)$ is true.

$f(x) = [x]$

(i) When $-1 \leq x < 0$, $[x] = -1$ Examining if $\lim_{x \rightarrow 0^-} f(x)$ exists

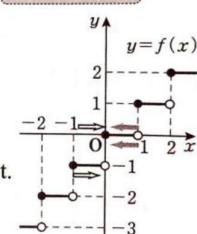
$\therefore \lim_{x \rightarrow 0^-} f(x) = -1$

(ii) When $0 \leq x < 1$, $[x] = 0$

$\therefore \lim_{x \rightarrow 0^+} f(x) = 0$

From (i) and (ii), $\lim_{x \rightarrow 0} f(x)$ does not exist.

Therefore, $f(x)$ is discontinuous at $x = 0$.



$$f(x) = \begin{cases} -ax^2 - 4a - 1 & (x \leq -2) \\ -ax - 9 & (x > -2) \end{cases}$$

If $f(x)$ is continuous for all x , then $f(x)$ must be continuous at $x = -2$. $f(-2) = \lim_{x \rightarrow -2^+} f(x)$ needs to be satisfied.

$$f(-2) = -a(-2)^2 - 4a - 1 = -8a - 1$$

$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} (-ax - 9) = 2a - 9$$

$$\therefore -8a - 1 = 2a - 9$$

$$\therefore a = \frac{4}{5}$$

As $x \rightarrow -2^+$,
 $f(x) = -ax - 9$

Intermediate Value Theorem

If $f(x)$ is continuous on the closed interval $[a, b]$ and $f(a) \neq f(b)$, then for any $k \in \mathbb{R}$ which lies between $f(a)$ and $f(b)$, the equation $f(x) = k$ has at least one real solution in the open interval (a, b) .

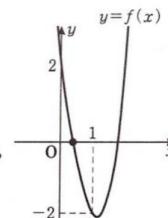
Prove that equation $x^4 - 5x + 2 = 0$ has at least one real solution in the interval $0 < x < 1$.

$$\text{Let } f(x) = x^4 - 5x + 2.$$

The function $f(x)$ is continuous on the $[0, 1]$.

$$\text{Also, } f(0) = 2 > 0, f(1) = -2 < 0$$

Therefore, from the Intermediate Value Theorem, $f(x) = 0$ has at least one real solution in the interval $0 < x < 1$.



N 141-160 : Differentiation

$f(x)$ is said to be differentiable at $x = a$ if $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists and is finite.

Properties of Derivatives

If $y = x^\alpha$, then $y' = \alpha x^{\alpha-1}$ (α is a real number)

If $y = kf(x)$, then $y' = kf'(x)$

If $y = f(x) \pm g(x)$, then $y' = f'(x) \pm g'(x)$

Product Rule

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

$$y = (x^2 - 2)(x^3 + 3x - 5)$$

$$\begin{aligned}y' &= (x^2 - 2)'(x^3 + 3x - 5) + (x^2 - 2)(x^3 + 3x - 5)' \\ &= 2x(x^3 + 3x - 5) + (x^2 - 2)(3x^2 + 3) \\ &= 5x^4 + 3x^2 - 10x - 6\end{aligned}$$

$$y = (3x + 2)(x - 1)(2x + 5) = (3x^2 - x - 2)(2x + 5)$$

$$\begin{aligned}y' &= (6x - 1)(2x + 5) + (3x^2 - x - 2) \cdot 2 \\ &= 18x^2 + 26x - 9\end{aligned}$$

Quotient Rule

$$\left[\frac{1}{g(x)} \right]' = -\frac{g'(x)}{[g(x)]^2}$$

$$\left[\frac{f(x)}{g(x)} \right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y = \frac{2x^2 - 3x + 5}{x^2 + 2}$$

$$y' = \frac{(4x-3)(x^2+2) - (2x^2-3x+5) \cdot 2x}{(x^2+2)^2} = \frac{3x^2-2x-6}{(x^2+2)^2}$$

$$y = \frac{x^2 - x + 1}{x^2 + x + 1}$$

$$y' = \frac{(2x-1)(x^2+x+1) - (x^2-x+1)(2x+1)}{(x^2+x+1)^2} = \frac{2x^2-2}{(x^2+x+1)^2}$$

Chain Rule

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

$$y = \frac{1}{(2-x)^3} = (2-x)^{-3} \quad y = \left(\frac{x}{3x+1}\right)^4$$

$$y' = -3(2-x)^{-4} \cdot (-1) \quad y' = 4\left(\frac{x}{3x+1}\right)^3 \cdot \frac{1 \cdot (3x+1) - x \cdot 3}{(3x+1)^2}$$

$$= \frac{3}{(2-x)^4} = \frac{4x^3}{(3x+1)^5}$$

$$y = (x + \sqrt{x^2+1})^4 = \left[x + (x^2+1)^{\frac{1}{2}}\right]^4$$

$$y' = 4 \left[x + (x^2+1)^{\frac{1}{2}}\right]^3 \cdot \left[1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x\right]$$

$$= 4(x + \sqrt{x^2+1})^3 \cdot \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}} \quad \left[x + (x^2+1)^{\frac{1}{2}}\right]' = 1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot (x^2+1)'$$

$$= \frac{4(x + \sqrt{x^2+1})^4}{\sqrt{x^2+1}}$$

$$y = x\sqrt{1+x^2} = x(1+x^2)^{\frac{1}{2}}$$

$$y' = 1 \cdot (1+x^2)^{\frac{1}{2}} + x \cdot \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x$$

$$= \sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}} = \frac{1+2x^2}{\sqrt{1+x^2}}$$

$$y = \frac{x}{\sqrt{x^2+1}} = x(x^2+1)^{-\frac{1}{2}}$$

$$y' = 1 \cdot (x^2+1)^{-\frac{1}{2}} + x \cdot \left(-\frac{1}{2}\right)(x^2+1)^{-\frac{3}{2}} \cdot 2x$$

$$= \frac{1}{\sqrt{x^2+1}} - \frac{x^2}{(x^2+1)\sqrt{x^2+1}} = \frac{1}{(x^2+1)\sqrt{x^2+1}}$$

N 161-170: Differentiation Trigonometric Functions

Derivatives of Trigonometric Functions

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x,$$

$$(\tan x)' = \frac{1}{\cos^2 x}$$

Important Note: All rules & properties (Product Rule, Quotient Rule, Chain Rule) apply.

$$y = (3x-2) \sin x$$

$$y' = (3x-2)' \sin x + (3x-2)(\sin x)'$$

$$= 3 \sin x + (3x-2) \cos x$$

$$y = \cos(3-2x^2)$$

$$y' = -\sin(3-2x^2) \cdot (3-2x^2)'$$

$$= 4x \sin(3-2x^2)$$

$$y = \cos^3(1-2x^2)$$

$$y' = 3\cos^2(1-2x^2) \cdot [\cos(1-2x^2)]'$$

$$= 3\cos^2(1-2x^2) \cdot [-\sin(1-2x^2)] \cdot (1-2x^2)'$$

$$= 12x \cos^2(1-2x^2) \sin(1-2x^2)$$

$$y = \frac{\tan x}{\cos x}$$

$$y' = \frac{(\tan x)' \cos x - \tan x (\cos x)'}{\cos^2 x}$$

$$= \frac{\frac{1}{\cos^2 x} \cdot \cos x + \frac{\sin x}{\cos x} \cdot \sin x}{\cos^2 x} = \frac{1 + \sin^2 x}{\cos^3 x}$$

$$y = \frac{1 - \sin x}{1 + \cos x}$$

$$y' = \frac{(1 - \sin x)'(1 + \cos x) - (1 - \sin x)(1 + \cos x)'}{(1 + \cos x)^2}$$

$$= \frac{-\cos x(1 + \cos x) + (1 - \sin x)\sin x}{(1 + \cos x)^2}$$

$$= \frac{-\cos x + \sin x - (\cos^2 x + \sin^2 x)}{(1 + \cos x)^2}$$

$$= -\frac{\cos x - \sin x + 1}{(1 + \cos x)^2}$$

N 171-180 : Differentiation of Logarithmic & Exponential Functions

Derivatives of Logarithmic Functions

$$(\ln|x|)' = \frac{1}{x}, \quad (\log_a|x|)' = \frac{1}{x \ln a}$$

$$y = \ln|x^2-3|$$

$$y = \log_2|1-2x|$$

$$y' = \frac{(x^2-3)'}{x^2-3}$$

$$= \frac{2x}{x^2-3}$$

$$y' = \frac{(1-2x)'}{(1-2x)\ln 2}$$

$$= -\frac{2}{(1-2x)\ln 2}$$

$$y = \ln(x + \sqrt{x^2+1})$$

$$y' = \frac{(x + \sqrt{x^2+1})'}{x + \sqrt{x^2+1}}$$

$$= \frac{1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x}{x + \sqrt{x^2+1}}$$

$$= \frac{\sqrt{x^2+1} + x}{(x + \sqrt{x^2+1})\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}$$

$$y = x(\ln x)^2$$

$$y' = (x)'(\ln x)^2 + x[(\ln x)^2]'$$

$$= 1 \cdot (\ln x)^2 + x \cdot 2 \ln x \cdot (\ln x)' = (\ln x)^2 + 2 \ln x$$

Useful Technique: Logarithmic Differentiation (appropriate when the function is a product/quotient of many factors)

$$y = \frac{(x+1)^3}{(x-2)^2(x+3)^4}$$

Taking the natural logarithm of the absolute values of both sides,

$$\ln|y| = \ln \left| \frac{(x+1)^3}{(x-2)^2(x+3)^4} \right|$$

$$= 3\ln|x+1| - (2\ln|x-2| + 4\ln|x+3|) \quad \leftarrow \begin{matrix} \ln MN \\ = \ln M + \ln N \end{matrix}$$

Differentiating both sides with respect to x ,

$$\frac{y'}{y} = \frac{3}{x+1} - \left(\frac{2}{x-2} + \frac{4}{x+3} \right) = -\frac{3x^2+x+16}{(x+1)(x-2)(x+3)}$$

$$\therefore y' = -\frac{3x^2+x+16}{(x+1)(x-2)(x+3)} \cdot \frac{(x+1)^3}{(x-2)^2(x+3)^4}$$

$$= -\frac{(3x^2+x+16)(x+1)^2}{(x-2)^3(x+3)^5}$$

Derivatives of Exponential Functions

$$(e^x)' = e^x, \quad (a^x)' = a^x \ln a \quad (a > 0, a \neq 1)$$

$$y = 5^{-x} \quad y = e^{\frac{1}{x}}$$

$$y' = 5^{-x} \ln 5 \cdot (-x)' \quad y' = e^{\frac{1}{x}} \cdot \left(\frac{1}{x} \right)' = -\frac{e^{\frac{1}{x}}}{x^2}$$

$$= -5^{-x} \ln 5$$

$$y = x e^{-x^2} \quad y = e^x \cos x$$

$$y' = (x)' e^{-x^2} + x(e^{-x^2})' \quad y' = (e^x)' \cos x + e^x (\cos x)'$$

$$= 1 \cdot e^{-x^2} - 2x^2 e^{-x^2} \quad = e^x \cos x - e^x \sin x$$

$$= (1-2x^2)e^{-x^2} \quad = e^x (\cos x - \sin x)$$

$$y = \frac{e^x - 1}{e^x + 1}$$

$$y' = \frac{(e^x - 1)'(e^x + 1) - (e^x - 1)(e^x + 1)'}{(e^x + 1)^2}$$

$$= \frac{e^x(e^x + 1) - (e^x - 1)e^x}{(e^x + 1)^2} = \frac{2e^x}{(e^x + 1)^2}$$

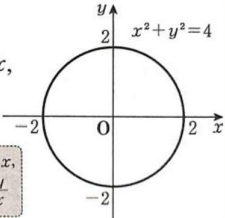
N 181-190 : Differentiation of Various Functions & Higher Order Derivatives

Implicit differentiation when the function is not expressed explicitly in terms of x .

$$x^2 + y^2 = 4$$

Differentiating both sides with respect to x ,

$$2x + 2y \cdot \frac{dy}{dx} = 0 \quad \leftarrow$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y} \quad \leftarrow \begin{matrix} \text{Differentiating } y^2 \text{ with respect to } x, \\ \frac{d}{dx} y^2 = \frac{d}{dy} y^2 \cdot \frac{dy}{dx} = 2y \cdot \frac{dy}{dx} \end{matrix}$$


Derivatives of Functions in Parametric Form

When $x = f(t)$ and $y = g(t)$,

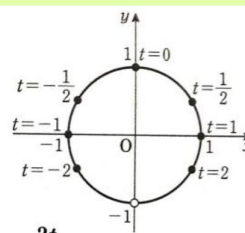
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$x = \frac{2t}{1+t^2}, \quad y = \frac{1-t^2}{1+t^2}$$

$$\frac{dx}{dt} = \frac{2(1+t^2) - 2t \cdot 2t}{(1+t^2)^2} = \frac{2(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{-2t(1+t^2) - (1-t^2) \cdot 2t}{(1+t^2)^2}$$

$$= -\frac{4t}{(1+t^2)^2} \quad \therefore \frac{dy}{dx} = -\frac{2t}{1-t^2}$$



The n^{th} order derivative of the function $f(x)$ is obtained by differentiating $f(x)$ for n times.

$$y = x^4 + 2x^3 - x + 1$$

$$y' = 4x^3 + 6x^2 - 1$$

$$\therefore y'' = 12x^2 + 12x$$

$$y''' = 24x + 12$$

Differentiating y'

Differentiating y''

N 191-200 : Various Properties of Derivatives

Recall that $f(x)$ is differentiable at $x = a$ if $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ exists and is finite.

Theorem: If $f(x)$ is differentiable at $x = a$, then it is continuous at $x = a$.

Rolle's Theorem

If the function $f(x)$ is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b) and $f(a) = f(b)$, then there exists at least one value c such that $f'(c) = 0$ and $a < c < b$.

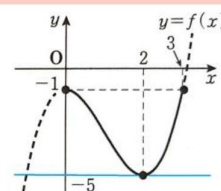
$$f(x) = x^3 - 3x^2 - 1 \quad [0, 3]$$

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

$$f'(c) = 3c(c-2) = 0$$

$$\text{Since } 0 < c < 3, c = 2 \quad \leftarrow$$

$c=0$ is not appropriate.



(Lagrange's) Mean Value Theorem

If the function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there exists at least one value c with $a < c < b$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$f(x) = x^3 - 3x^2 + 2x \quad [0, 3]$$

$$\frac{f(3) - f(0)}{3 - 0} = \frac{6 - 0}{3} = 2 \quad \leftarrow a=0, b=3$$

$$f'(x) = 3x^2 - 6x + 2$$

From the Mean Value Theorem, there exists at least one value c such that

$$2 = 3c^2 - 6c + 2 \text{ and } 0 < c < 3.$$

$$3c(c-2) = 0$$

$$\text{Since } 0 < c < 3, c = 2 \quad \leftarrow$$

$c=0$ is not appropriate.

